DEFORMATIONAL ANISOTROPY

(O DEFORMATSIONNOI ANIZOTROPII)

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In this paper deformational anisotropy is meant to be that anisotropy which develops in a body as a result of its elastic deformation. Considered is the problem of the state of stress in a body subjected, from a "natural" state, to two consecutive elastic deformations. It is shown, that if the first deformation is homogeneous and different from a hydrostatic tension or compression, then with the application of a second deformation the body deforms in general as an orthotropic one; the principal directions of elasticity coincide with the principal axes of the homogeneous deformation, and the elastic constants of this body can be expressed in terms of the elastic constants of the original isotropic body and its principal elongations in first deformation. Thus, in solving this problem, the possibility is opened to use the theory of elasticity of anisotropic bodies. As applications, torsion of an elongated bar and bending of an extended and sheared plate are considered.

1. Notation

Before and after its deformation, the body is referred to three fixed and mutually perpendicular axes 1,2 and 3. The coordinates of any point of the body before the first deformation (in the "natural" state) are designated by x_i(here, and in what follows, $i = 1,2,3$). The coordinates of this point after the first deformation (in the "initial" state, since this is what it is for the second deformation) are designated by y_i. The first deformation is characterized by displacements u_i, such that $y_i = x_i + u_i$ (Fig.1). The components of the first deformation are determined by formulas [1,2].

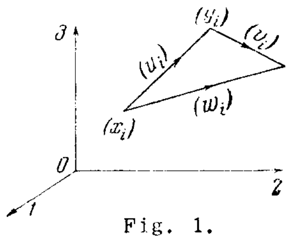

Fig. 1.

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$$e_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \sum_r \frac{\partial u_r}{\partial x_i}\frac{\partial u_r}{\partial x_k} \qquad (i,\ k,\ r = 1,\ 2,\ 3) \tag{1.1}$$

The second deformation is characterized by displacements of points of the body after the first deformation. The components of the second deformation are determined by formulas

$$f_{ik} = \frac{\partial v_i}{\partial y_k} + \frac{\partial v_k}{\partial y_i} + \sum_r \frac{\partial v_r}{\partial y_i}\frac{\partial v_r}{\partial y_k} \qquad (i,\ k,\ r = 1,\ 2,\ 3) \tag{1.2}$$

The total deformation, as a result of the first and second deformations, is given by displacements $w_i = u_i + v_i$ of points of a body from their position in the natural state. The components of the total deformation are defined by formulas

$$g_{ik} = \frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} + \sum_r \frac{\partial w_r}{\partial x_i}\frac{\partial w_r}{\partial x} \qquad (i,\ k,\ r = 1,\ 2,\ 3) \tag{1.3}$$

The components of stress after two deformations are $\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}''$.

Here σ_{ij}' indicate the stresses corresponding to displacements u_i. These are called "initial" stresses, since this is what they are for the second deformation; σ_{ij}'' indicate stresses corresponding to displacements v_i. These shall be called "secondary" stresses.

2. First deformation

The first deformation is assumed to be homogeneous and pure. Generally in this case, the displacements of a point of the body will be $u_1 = \alpha_1 x_1$, $u_2 = \alpha_2 x_2$, $u_3 = \alpha_3 x_3$, if the axes of coordinates 1,2,3 (see Section 1) are directed parallel with the principal axes of deformation. The coefficients $\alpha_i = \partial u_i / \partial x_i$ are the principal elongations. The coordinates of the point after deformation and the principal components of strain will be, respectively

$$\begin{gathered} y_1 = (1 + \alpha_1) x_1, \qquad y_2 = (1 + \alpha_2) x_2, \qquad y_3 = (1 + \alpha_3) x_3 \\ e_1 = 2\alpha_1 + \alpha_1^2, \qquad e_2 = 2\alpha_2 + \alpha_2^2, \qquad e_3 = 2\alpha_3 + \alpha_3^2 \end{gathered} \tag{2.1}$$

The retention of quadratic terms, in the last expressions above, indicates that the first deformation is not asumed to be small, as in the linear theory of elasticity; thus, on the basis of certain considerations, it is necessary to take second-order effects into account in this deformation.

The invariants of strain are given by

$$J_n = e_1^n + e_2^n + e_3^n \qquad (n = 1,\ 2,\ 3,\ ..) \tag{2.2}$$

derived from expressions

$$J_1 = e_{11} + e_{22} + e_{33}, \qquad J_2 = e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{23}^2 + 2e_{31}^2 + 2e_{12}^2 \qquad (2.3)$$
$$J_3 = e_{11}^3 + e_{22}^3 + e_{33}^3 + 3e_{11}(e_{12}^2 + e_{31}^2) + 3e_{22}(e_{12}^2 + e_{23}^2) +$$
$$+ 3e_{33}(e_{31}^2 + e_{23}^2) + 6e_{23}e_{31}e_{12}$$

in which the components of strain with different subscripts are set equal to zero, and e_{ii} is denoted by e_i.

In the linear theory of elasticity, the principal stresses and the elastic potential (density of the potential strain energy Φ, referred to a unit volume before deformation) are related by

$$\sigma = \lambda\theta + 2\mu\alpha_i, \qquad \Phi = \frac{\lambda}{8} J_1^2 + \frac{\mu}{4} J_2, \qquad \sigma_i = 2\frac{\partial\Phi}{\partial e_i} = \frac{\partial\Phi}{\partial\alpha_i} \qquad (2.4)$$

where $\theta = \alpha_1 + \alpha_2 + \alpha_3$ and the strain invariants are given by (2.2), where the components of strain (2.1) are taken without quadratic terms.

Retaining these quadratic terms, we shall determine the principal stresses corresponding to displacements $u_i = \alpha_i x_i$, i.e. initial stresses (see Section 1); the stresses σ_i will be taken per unit area of the deformed body, by the formulas

$$\sigma_1' = (\lambda\theta + 2\mu\alpha_1)/[(1 + \alpha_2)(1 + \alpha_3)] = (\lambda\theta + 2\mu)(1 - \alpha_2 - \alpha_3) \qquad (2.5)$$
$$\sigma_2' = (\lambda\theta + 2\mu\alpha_2)(1 - \alpha_3 - \alpha_1), \qquad \sigma_3' = (\lambda\theta + 2\mu\alpha_3)(1 - \alpha_1 - \alpha_2)$$

with an accuracy including squares of the principal elongations. In contrast to (2.4), the relationship between the principal stresses and the elastic potential Φ_1, taken per unit volume of the body before its deformation, is expressed in the non-linear theory of elasticity by formulas [1]

$$\sigma_i' = \frac{1}{\Delta}\left(\frac{\partial y_i}{\partial x_i}\right)^2 2\frac{\partial\Phi_1}{\partial e} \qquad (2.6)$$

where $\Delta = D(y_1, y_2, y_3) / D(x_1, x_2, x_3)$ is the ratio of the volume elements of the body before and after its deformation, and y_i is given in the case considered by formulas (2.1). Comparing (2.6) and (2.4), and observing that, because of (2.1), $\partial/\partial\alpha_1 = 2(1 + \alpha_i)\,\partial/\partial e_i$ we find

$$\frac{\partial\Phi_1}{\partial\alpha_1} = \lambda\theta + 2\mu\alpha_1, \qquad \frac{\partial\Phi_1}{\partial\alpha_2} = \lambda\theta + 2\mu\alpha_2, \qquad \frac{\partial\Phi_1}{\partial\alpha_3} = \lambda\theta + 2\mu\alpha_3 \qquad (2.7)$$

Integrating equations (2.7), we obtain an expression for the elastic potential of the first deformation in terms of the principal elongations:

$$\Phi_1 = \frac{1}{2}(\lambda + 2\mu)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \lambda(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \qquad (2.8)$$

To find an expression for this potential in terms of invariants of the first deformation, we expand the expression $\alpha_i = \sqrt{(1 + e_i)} - 1$, obtained from (2.1), into a series of powers of e_i, converging for $|e_i| < 1$.

Taking only the first two terms of this series, i.e. setting

$$\alpha_i = \frac{1}{2} e_i - \frac{1}{8} e_i^2 \tag{2.9}$$

and substituting this expression into (2.8), we obtain

$$\Phi_1 = \frac{\lambda}{8} J_1^2 + \frac{\mu}{4} J_2 - \frac{\lambda}{16} J_1 J_2 - \frac{\mu}{8} J_3 \tag{2.10}$$

Here the strain invariants are taken from (2.2), and the principal components of strain from (2.1). Because of the condition $|e_i| < 1$, it follows from (2.9), that the elastic potential (2.10) and the formulas (2.5) corresponding to it are applicable to such deformations of isotropic bodies, for which the principal elongations $\alpha_i < 0.375$.

The elastic potential (2.10) may be compared with the one of the "five-constant" theory developed by Murnaghan [3,4]:

$$\Phi' = \frac{1}{2} (\lambda + 2\mu) \varepsilon_I - 2\mu\varepsilon_{II} + l\varepsilon_I^3 + m\varepsilon_I\varepsilon_{II} + n\varepsilon_{III} \tag{2.11}$$

Here l, m, n are the new (in addition to λ, μ) elastic constants, which should be determined from experiments or theoretical considerations. The elastic potential (2.11) is written in terms of Eulerian variables, i.e. the coordinates y of the points of the body after its deformation are taken as independent variables, in contrast to formulas (1.1). The invariants of strain ϵ_I, ϵ_{II}, ϵ_{III} are also taken in a form different from the one adopted by us in (2.3). It can be shown that if written in terms of Lagrangian variables (the independent variables are the coordinates x_i of the points of the body before its deformation) and in terms of invariants given by (2.3), the elastic potential (2.11) will be:

$$\Phi' = \frac{\lambda}{8} J_1^2 + \frac{\mu}{4} J_2 - \left(\frac{\lambda}{4} + \frac{m}{16} + \frac{n}{16}\right) J_1 J_2 + \left(\frac{l}{8} + \frac{m}{16} + \frac{n}{48}\right) J_1^3 + \left(\frac{n}{24} - \frac{\mu}{2}\right) J_3 \tag{2.12}$$

Comparing this expression with (2.10), we find

$$l = {}^3/_2\lambda + 3\mu, \quad m = -(3\lambda + 9\mu), \quad n = 9\mu \tag{2.13}$$

Consequently, with such values of the elastic constants, the elastic potential of the five-constant theory coincides with ours. It is of interest to note that N.V. Zvolinskii and P.M. Riz [5] basing their work on different considerations, give values of the elastic constants in the five-constant theory which coincide with (2.13).

3. Application of deformation

Assume that the first elastic deformation is followed by a second, also elastic one. Designating, as in Section 1, by f_{ik} and g_{ik} the components of the second and total strain, we find the following components from (1.3), taking into account (1.1), (1.2) and (2.1)

$$g_{ii} = 2\alpha_i + \alpha_i^2 + (1+\alpha_i)^2 f_{ii}, \qquad g_{ik} = (1+\alpha_i)(1+\alpha_k) f_{ik} \tag{3.1}$$

Substituting into (2.10) in place of the invariants of the first deformation the invariants of the total deformation, obtained from (2.3) by the exchange of e_{ik} to g_{ik}, and dividing the result by $(1+\alpha_1)$ $(1+\alpha_2)$ $(1+\alpha_3)$, we arrive at the elastic potential of the total deformation referred to the initial state

$$\Phi = \Phi_1^* + \Phi_{12} + \Phi_2 \tag{3.2}$$

where

$$\Phi_1^* = \frac{1}{[(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)]}\left[\frac{\lambda}{8}J_1'^2 + \frac{\mu}{4}I_2' - \frac{\lambda}{16}J_2'J_2' - \frac{\mu}{8}J_3'\right] u$$

The invariants of the first deformation are denoted by primes,

$$\Phi_{12} = K_1 f_{11} + K_2 f_{22} + K_3 f_{33}$$

$$\Phi_2 = L_1 f_{11}^2 + L_2 f_{22}^2 + L_3 f_{33}^2 + M_1 f_{22} f_{33} + M_2 f_{33} f_{11} + M_3 f_{11} f_{22} + \\ + N_1 f_{23}^2 + N_2 f_{31} + N_3 f_{12}^2$$

$$K_1 = \frac{1+\alpha_1}{(1+\alpha_2)(1+\alpha_3)}\left[\frac{\lambda}{4}J_1' + \frac{\mu}{2}e_1 + \frac{\lambda}{16}(J_2' + 2J_1'e_1) + \frac{3}{8}\mu e_1^2\right]$$

$$L_1 = \frac{(1+\alpha_1)^3}{(1+\alpha_2)(1+\alpha_3)}\left[\frac{\lambda+2\mu}{8} - \frac{\lambda}{16}J_1' - \frac{\lambda+3\mu}{8}e_1\right]$$

$$M_1 = \frac{(1+\alpha_2)(1+\alpha_3)}{1+\alpha_1}\left[\frac{\lambda}{4} + \frac{\lambda}{8}(e_2+e_3)\right]$$

$$N_1 = \frac{(1+\alpha_2)(1+\alpha_3)}{1+\alpha_1}\left[\frac{\mu}{2} - \frac{\lambda}{8}J_1' - \frac{3}{8}\mu(e_2+e_3)\right]$$

K_2, K_3; L_2, L_3; M_2, M_3; N_2, N_3 are derived from K_1, L_1, M_1, N_1 respectively by a circular permutation of subscripts of α and e.

To simplify the calculations, we omitted the terms higher than those of the second power in f, i.e. we considered the second deformation as being small; therefore, f in (3.2) should be taken from (1.2) without the quadratic terms. $\Phi_1{}^*$ is the elastic potential of the first deformation referred, as also other terms in (3.2), to a unit volume in the initial state of the body. Φ_{12} expresses, as can be shown, the "partial" work of the initial stresses in the second deformation. Φ_2, representing the elastic potential of the second deformation, also appears as the elastic potential of an orthotropic body [6], whose principal directions of elasticity coincide with the principal axes of the first deformation. This indicates that, by the application of a second deformation, the body deforms as an orthotropic one; that is, as a result of the first deformation it becomes orthotropic. Therefore, the stresses produced by displacements v_i of the second deformation (secondary stresses) may be taken from the corresponding solutions of the theory of elasticity of anisotropic bodies. It will be our problem to find expressions for the elastic constants of this orthotropic body.

Remark 1. If the first deformation were not homogeneous and pure, then the deformed body would be orthotropic at each of its points (i.e. in the infinitely small vicinity of such points); however, the principal directions of elasticity and the elastic constants at various points of the body, in general, would be different. The principal elongations α_i would be functions of the coordinates of the points of the body, and the displacements $u_i = \alpha_i x_i$ would define the deformation at point (x_i), only with an accuracy within a rotation of its vicinity around this point.

Remark 2. The statement, that Φ_2 represents the elastic potential of the second deformation needs an explanation. During the second deformation, besides the forces producing it, there are also present the forces which produced the first deformation (we shall call them continuously acting forces). Therefore, to the density of the energy of deformation -- to the elastic potential Φ (3.2) - - should be added the density Ψ of that potential energy which corresponds to those forces (that is the mechanical energy of the system associated with the first deformation); and the secondary stresses are expressed as derivatives of the sum $\Phi + \Psi$ with respect to components f of the second deformation. However, it is possible to show that

$$\frac{\partial(\Phi + \Psi)}{\partial f_{ik}} = \frac{\partial(\Phi_1{}^* + \Phi_{12} + \Phi_2 + \Psi)}{\partial f_{ik}} = \frac{\partial \Phi_2}{\partial f_{ik}} \tag{3.3}$$

For this reason Φ_2 was called the elastic potential of the second deformation. Indeed, as it was shown, Φ_{12} represents the specific work of initial stresses in the second deformation. Since this work is equal to the specific work of the continuously acting forces in this same deformation, so the density of the potential energy Ψ corresponding to them, decreases if this work is positive, and increases if the work is negative. Assuming Ψ as being equal to zero in the initial state (without loss of generality), we obtain $\Psi = -\Phi_{12}$, from which equation (3.3) follows. The absence, in the sum $\Phi + \Psi$, as a consequence of $\Psi = -\Phi_{12}$, of first power terms in f, expresses the stability of the initial state of the body relative to deformation.

If the second deformation is not small, then $\Psi \neq -\Phi_{12}$. Not having the possibility of entering into details, we merely mention that in such cases we can expand Ψ into power series of the components f of the second deformation. Then, in the sum $\Phi + \Psi$, due to stability of the initial state, Φ_{12} will reduce by the first-power terms of f entering into Ψ, and terms with higher powers of f are added to Φ_2. Moreover, the additional terms in Φ_2, appear also in (3.2), since f cannot be taken from (1.2) without quadratic terms, as it happened in the derivation of (3.2).

Remark 3. The question relating to the type of anisotropy, developed as a consequence of an elastic deformation, was considered apparently first by Voigt [13]. He came to the conclusion that after a large deformation (to which subsequently a small one is applied) the body attains an elastic symmetry of a rhombic crystal if the first deformation is homogeneous, and of a hexagonal crystal if it is a uni-axial elongation. However, making use of the five-constant elastic potential suggested by him, Voigt takes the components of the deformation (which he does not consider small) without the quadratic terms, and the relation between the elastic potential and the stresses taken on the same form, as in the linear theory of elasticity. In our deductions above, we arrived by a different method to a result qualitatively equivalent to Voigt's conclusion, if it is taken into account that the elastic potential and the generalized Hooke's law for rhombic and hexagonal crystals, have the same form as for the orthotropic and transversly isotropic bodies, respectively.

4. Elastic constants of an orthotropic body

Hooke's law for an orthotropic body,

$$\begin{aligned} \sigma_{11} &= \tfrac{1}{2}(A_{11}f_{11} + A_{12}f_{22} + A_{13}f_{33}), & \sigma_{23} &= A_{44}f_{23} \\ \sigma_{22} &= \tfrac{1}{2}(A_{12}f_{11} + A_{22}f_{22} + A_{23}f_{33}), & \sigma_{31} &= A_{55}f_{31} \\ \sigma_{33} &= \tfrac{1}{2}(A_{13}f_{11} + A_{23}f_{22} + A_{33}f_{33}), & \sigma_{12} &= A_{66}f_{12} \end{aligned} \tag{4.1}$$

contains the "moduli of elasticity" A_{ik}, which can be expressed in terms of the "components of strain" a_{ik}

$$\begin{aligned} A_{11} &= \frac{a_{22}a_{33} - a_{23}^2}{\Delta}, & A_{23} = A_{32} &= \frac{a_{12}a_{13} - a_{11}a_{13}}{\Delta}, & A_{44} &= \frac{1}{a_{44}} \\ A_{22} &= \frac{a_{33}a_{11} - a_{13}^2}{\Delta}, & A_{31} = A_{13} &= \frac{a_{12}a_{23} - a_{13}a_{22}}{\Delta}, & A_{55} &= \frac{1}{a_{55}} \\ A_{33} &= \frac{a_{11}a_{22} - a_{12}^2}{\Delta}, & A_{12} = A_{21} &= \frac{a_{31}a_{32} - a_{12}a_{33}}{\Delta}, & A_{66} &= \frac{1}{a_{66}} \end{aligned} \tag{4.2}$$

$$\Delta = a_{11}a_{22}a_{33} - 2a_{12}a_{13}a_{23} - a_{12}^2a_{33} - a_{13}^2a_{22} - a_{23}^2a_{11}$$

As is known, the coefficients of deformation are related to the so-called "technical constants" of an orthotropic body by the relation [7]

$$\begin{aligned} &a_{11} = \frac{1}{E_1}, \quad a_{22} = \frac{1}{E_2}, \quad a_{33} = \frac{1}{E_3}, \quad a_{44} = \frac{1}{G_{23}}, \quad a_{55} = \frac{1}{G_{13}}, \quad a_{66} = \frac{1}{G_{12}} \\ &a_{12} = -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1}, \quad a_{13} = -\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1}, \quad a_{23} = -\frac{\nu_{32}}{E_3} = -\frac{\nu_{23}}{E_1} \end{aligned} \tag{4.3}$$

where E_i, G_{ik}, ν_{ik} are Young's moduli, shear moduli and Poisson's ratios, respectively. Introducing (4.1) in the well-known expression for the elastic potential, we obtain

$$\Phi = \tfrac{1}{2}(\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{23}e_{23} + \sigma_{31}e_{31} + \sigma_{12}e_{12}) \tag{4.4}$$

after replacing in it e_{ii}, e_{ik} by $\frac{1}{2}f_{ii}$, f_{ik}. Comparing the derived equation with Φ_2 in (3.2), we obtain an expression for the moduli of elasticity A_{ik} in terms of λ, μ, α, and further after a substitution

$$\lambda = \frac{E_0 \nu_0}{(1+\nu_0)(1-2\nu_0)}, \qquad \mu = \frac{E_0}{2(1+\nu_0)} \tag{4.5}$$

in terms of E_0, ν_0, α_i. Substituting the derived expression for the moduli of elasticity A_{ik} and expressions (4.3) into (4.2), we obtain equations, from which the technical constants of an orthotropic body may be expressed in terms of Young's modulus E_0 and Poisson's ratio ν_0, of an initially isotropic body, and by its principal elongations in a homogeneous deformation. Omitting all intermediate calculations, we give the expressions for these technical constants with an accuracy within the second power of elongations α_i

$$\begin{aligned} E_1 = E_0 \Big[1 &- \frac{2\nu_0^2}{1-2\nu_0}\alpha_1 - \frac{1-\nu_0}{1-2\nu_0}(\alpha_2+\alpha_3) - \frac{15-15\nu_0-12\nu_0^2+6\nu_0^3+4\nu_0^4}{2(1+\nu_0)(1-2\nu_0)}\alpha_1^2 + \\ &+ \frac{2-\nu_0-11\nu_0^2+8\nu_0^3}{2(1+\nu_0)(1-2\nu_0)}(\alpha_2^2+\alpha_3^2) - \frac{3\nu_0-3\nu_0^2+2\nu_0^3}{(1+\nu_0)(1-2\nu_0)}\alpha_1(\alpha_2+\alpha_3) + \\ &+ \frac{1+\nu_0-8\nu_0^3}{(1+\nu_0)(1-2\nu_0)}\alpha_2\alpha_3 \Big] \end{aligned} \tag{4.6}$$

(E_2 and E_3 are obtained by cyclic permutation of subscripts of α);

$$\begin{aligned} \nu_{12} = \nu_0 \Big[1 &+ (1+\nu_0)\alpha_1 - \frac{3-9\nu_0+4\nu_0^2+4\nu_0^3}{2(1-2\nu_0)}\alpha_1^2 + \frac{6-12\nu_0+4\nu_0^2}{1-2\nu_0}\alpha_2^2 - \\ &- \frac{4\nu_0(1-\nu_0)}{1-2\nu_0}\alpha^2_3 - \frac{1-4\nu_0+3\nu_0^2}{1-2\nu_0}\alpha_1\alpha_2 + \frac{\nu_0-3\nu_0^2}{1-2\nu_0} + 4\nu_0\alpha_2\alpha_3 \Big] \end{aligned} \tag{4.7}$$

(ν_{23} and ν_{31} are obtained by cyclic permutation; ν_{ki} are obtained from ν_{ik} by an interchange of subscripts i and k;

$$\begin{aligned} G_{12} = G_0 \Big[1 &- \frac{1-\nu_0}{1-2\nu_0}\alpha_3 - \frac{1}{2(1-2\nu_0)}(\alpha_1+\alpha_2) - \frac{3(3-4\nu_0)}{4(1-2\nu_0)}(\alpha_1^2+\alpha_2^2) + \\ &+ \frac{2-3\nu_0}{2(1-2\nu_0)}\alpha_3^2 - \frac{2(1-\nu_0)}{1-2\nu_0}\alpha_1\alpha_2 + \frac{1}{2}(\alpha_1+\alpha_2)\alpha_3 \Big] \end{aligned} \tag{4.8}$$

(for G_{23} and G_{31} again cyclic permutation has to be applied).

5. Special cases

In formulas expressing the initial stresses (2.5), and in formulas (4.6) to (4.8) defining the elastic constants of an orthotropic body, produced from an isotropic body after the first homogeneous deformation, the principal elongations α_i in this deformation are presumed to be independent. Cases are given below, when they are connected by some relationships.

1. *Hydrostatic tension or compression.* In this case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$; $\alpha > 0$ for tension. As in equations (4.6) to (4.8) all quantities E_i, G_{ik} and ν_{ik} differ one from another only by the arrangement of subscripts of α, Young's moduli, shear moduli and Poisson's ratios of an orthotropic body will be alike, that is the isotropy of the body will not be disturbed by a hydrostatic tension or compression. This deformation will

merely alter numerically the original values of Young's modulus and Poisson's ratio, which they had before deformation and which will be now

$$E = E_0\left[1 - \frac{2(1-\nu_0+\nu_0^2)}{1-2\nu_0}\alpha - \frac{9-12\nu_0+10\nu_0^2+4\nu_0^3}{2(1-2\nu_0)}\alpha^2\right]$$
$$\nu = \nu_0\left[1+(1+\nu_0)\alpha + \frac{1}{2}\,(7+9\nu_0+2\nu_0^2)\alpha^2\right] \tag{5.1}$$

The initial stresses from (2.5) will be

$$\sigma_1' = \sigma_2' = \sigma_3' = p = 3k\alpha - 6k\alpha^{\cdot\cdot} \tag{5.2}$$

where $k = (3\lambda + 2\mu)/3$ is the bulk modulus.

2. *Uniaxial extension or compression.* An isotropic, prismatic bar, after being extended or compressed ($\alpha < 0$), becomes transversely isotropic with the planes of isotropy perpendicular to its axis. Assuming that the latter coincides with the axis of the coordinate 3 and by putting $\alpha_3 = \alpha$, $\alpha_1 = \alpha_2 = -\nu_0\alpha$, we derive from (4.6) to (4.8) the values of its technical constants (in the notation, adopted in reference [7]):

$$E = E_1 = E_2 = E_0\left[1-(1+\nu_0^2)\alpha + \frac{2-5\nu_0-15\nu_0^2+25\nu_0^3+2\nu_0^4-4\nu_0^5}{2(1-2\nu_0)}\alpha^2\right]$$
$$\nu = \nu_{12} = \nu_0\left|1-\nu_0(1+\nu_0)\alpha - \frac{\nu_0(1+\nu_0)(8-13\nu_0-2\nu_0^2+4\nu_0^3)}{2(1-2\nu_0)}\alpha^2\right]$$
$$E' = E_3 = E_0\left[1+2\nu_0\alpha - \frac{15-30\nu_0+18\nu_0^3}{2(1-2\nu_0)}\alpha^2\right] \tag{5.3}$$
$$\nu' = \nu_{32} = \nu_{31} = \nu_0\Big]1+(1+\nu_0)\alpha - \tfrac{1}{2}(3-5\nu_0-3\nu_0^2)\alpha^2\Big]$$
$$G' = G_{23} = G_{31} = G_0\left[1-\frac{1}{2}(1-\nu_0)\alpha - \frac{9-18\nu_0+7\nu_0^3-2\nu_0^3}{4(1-2\nu_0)}\alpha^2\right]$$

The initial stresses from (2.5) will be

$$\sigma_1' = \sigma_2' = 0, \quad \sigma_3' = E_0\alpha(1+2\nu_0\alpha) \tag{5.4}$$

3. *Pure shear*. If the principal elongations α_1 and α_2 satisfy the condition $(1+\alpha_1)(1+\alpha_2) = 1$ and $\alpha_3 = 0$, then a deformation field of pure shear will be produced relative to axes 1,2,3. Suppose that compression takes place parallel to axis 1 and extension parallel to axis 2. We put [6]

$$\alpha_1 = \sec\beta - \mathrm{tg}\,\beta - 1, \qquad \alpha_2 = \sec\beta + \tan\beta - 1 \tag{5.5}$$

These expressions for principal elongations satisfy the set condition. Letting $\tan\beta = (\alpha_2 - \alpha_1)/2 = s/2$, we derive

$$\alpha_1 = \tfrac{1}{2}\sqrt{4+s^2} - \tfrac{1}{2}s - 1 \approx -\tfrac{1}{2}s\left(1-\tfrac{1}{4}s\right)$$
$$\alpha_2 = \tfrac{1}{2}\sqrt{4+s^3} + \tfrac{1}{2}s - 1 \approx \tfrac{1}{2}s\left(1+\tfrac{1}{4}s\right) \tag{5.6}$$

with an accuracy to s^2 inclusively. If a plate with its edges parallel to the axes 1 and 2 is subjected to pure shear, then it becomes orthotropic and its technical constants are determined by substituting the value of α previously derived into expressions (4.6) to (4.8). Among these technical constants we present the following:

$$
\begin{aligned}
E_1 &= E_0\left[1 - \frac{1+\nu_0}{2}s - \frac{7-10\nu_0+3\nu_0^2-2\nu_0^3+2\nu_0^4}{4(1-2\nu_0)(1+\nu_0)}s^2\right]\\
E_2 &= E_0\left[1 + \frac{1+\nu_0}{2}s - \frac{7-10\nu_0+3\nu_0^2-2\nu_0^3+2\nu_0^4}{4(1-2\nu_0)(1+\nu_0)}s^2\right]\\
\nu_1 = \nu_{12} &= \nu_0\left[1 - \frac{1+\nu_0}{2}s - \frac{3-6\nu_0+2\nu_0^2-\nu_0^3}{2(1-2\nu_0)}s^2\right]\\
\nu_2 = \nu_{21} &= \nu_0\left[1 + \frac{1+\nu_0}{2}s - \frac{3-6\nu_0+2\nu_0^2-\nu_0^3}{2(1-2\nu_0)}s^2\right]\\
G = G_{12} &= G_0\left[1 - \frac{3-4\nu_0}{4(1-2\nu_0)}s^2\right]
\end{aligned}
\tag{5.7}
$$

In above formulas $s = \alpha_2 - \alpha_1$ The initial stresses, by (2.5), will be

$$
\sigma_1' = -\mu s + \frac{\lambda+3\mu}{4}s^2, \qquad \sigma_2' = \mu s + \frac{\lambda+3\mu}{4}s^2, \qquad \sigma_3' = \frac{\lambda}{4}s^2 \tag{5.8}
$$

We note that on axes 1′, 2′, 3, produced by the rotation of axes 1, 2 through an angle $\phi = \pi/4 - \beta/2$ around the axis 3 (see Fig.2), the principal elongations (5.6) define the deformation field of simple shear, and s is the "magnitude" of this shear, i.e. the displacement of any point (x_i), referred to axes 1′, 2′, 3, will be $u_i' = sx_2'$, $u_2' = u_3 = 0$.

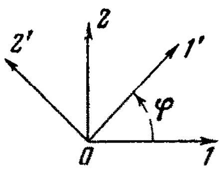

Fig. 2.

4. *State of plane stress*. For $\sigma_3' = 0$

$$
\alpha_3 = -\frac{\lambda}{\lambda+2\mu}(\alpha_1+\alpha_2) = -\frac{\nu_0}{1-\nu_0}(\alpha_1+\alpha_2) \tag{5.9}
$$

The initial stresses different from zero will be, from (2.9)

$$
\begin{aligned}
\sigma_1' &= \frac{E_0}{1-\nu_0^2}(\alpha_1+\nu_0\alpha_2) + \frac{\nu_0}{1-\nu_0}\alpha_1^2 - \frac{\nu_0(1-2\nu_0)}{1-\nu_0}\alpha_2^2 - \frac{1-2\nu_0-\nu_0^2}{1-\nu_0}\alpha_1\alpha_2\\
\sigma_2' &= \frac{E_0}{1-\nu_0^2}(\nu_0\alpha_1+\alpha_2) - \frac{\nu_0(1-2\nu_0)}{1-\nu_0}\alpha_1^2 + \frac{\nu_0}{1-\nu_0}\alpha_2^2 - \frac{1-2\nu_0-\nu_0^2}{1-\nu_0}\alpha_1\alpha_2,
\end{aligned}
\tag{5.10}
$$

6. Some applications

1. *Torsion of an extended or compressed bar*. Let the relative elongation of the axis of the bar, coinciding with the coordinate axis 3, be equal to α. The initial stresses may be found from (5.4). The bar will be twisted as a transversely isotropic one, with the elastic constants from (5.3), due to the action of a tensile force $p = sE_0\alpha(1 + 2\nu_0\alpha)$; where s is the area of the cross-section after elongation. However, in the case of a transversely isotropic rod, in first approximation, the

tensile force does not affect the magnitude and the distribution of shear stresses, and the latter is equal to the one in an isotropic bar, for given twisting moment M_t [7]. Therefore, the non-vanishing secondary stresses in a bar, whose cross-section before elongation is bounded, for example, by a contour $x_1^2/a^2 + x_2^2/b^2 = 1$, will be

$$\sigma''_{13} = -\frac{2M_t x_2}{\pi a b^3 (1-\nu_0\alpha)^3} \approx -\frac{2M_t x_2}{\pi a b^3}(1 + 3\nu_0\alpha + 6\nu_0^2\alpha^2)$$
$$\sigma''_{23} = \frac{2M_1 x_1}{\pi a^3 b (1-\nu_0\alpha)^3} \approx \frac{2M_t x_1}{\pi a^3 b}(1 + 3\nu_0\alpha + 6\nu_0^2\alpha^2) \quad (6.1)$$

It is taken into account here that after elongation of the bar, its cross-section will be bounded by an ellipse $y_1^2/[a^2(1-\nu_0\alpha)^2] + y_2^2/[b^2(1-\nu_0\alpha)^2] = 1$. The torsional rigidity of an extended or compressed bar will be

$$G = \frac{\pi a^3 b^3}{a^2+b^2}(1-\nu_0\alpha)^4 G' \approx G_t^\circ\left[1 - \frac{1+7\nu_0}{2}\alpha - \frac{9-26\nu_0+7\nu_0^2+30\nu_0^3}{4(1-2\nu_0)}\alpha^2\right] \quad (6.2)$$

where G' is given by (5.3) and G_t° denotes the rigidity of a bar that is neither elongated nor compressed.

2. *Bending of an extended or compressed plate*. A rectangular anisotropic plate, extended (compressed) in two directions parallel to its edges, will bend under an arbitrary, normal load, as an orthotropic plate with its principal directions parallel to the edges, under the action of forces $p_1 = \sigma_1' h$, $p_2 = \sigma_2' h$, uniformly distributed along these edges, and taken per unit length, where σ_1' and σ_2' are given by (5.10), and the thickness of the plate from (5.9) is

$$h = h_0\left[1 - \frac{\nu_0}{1-\nu_0}(\alpha_1+\alpha_2)\right] \quad (6.3)$$

if h_0 is the thickness of the plate before elongation, and α_1 and α_2 are its relative elongations. The theory of bending of orthotropic plates is well developed (see references [8-11]). The bending and torsional rigidities, required for the analysis of the plate by this theory,

$$D_1 = E_1 h^3/12(1-\nu_1\nu_2), \quad D_2 = E_2 h^3/12(1-\nu_1\nu_2)$$
$$D_k = Gh^3/12, \quad D_3 = D_2\nu_1 + 2D_k \quad (6.4)$$

are found with the aid of formulas (4.6) to (4.8), putting in them

$$\nu_{12} = \nu_1, \quad \nu_{21} = \nu_2, \quad G_{12} = G, \quad \alpha_3 = -\nu_0(\alpha_1+\alpha_2)/(1-\nu_0) \quad (6.5)$$

$$D_1 = \frac{E_0 h_0^3}{12(1-\nu_0^2)}\left[1 - \frac{2\nu_0}{1-\nu_0}\alpha_1 - \frac{1+2\nu_0-\nu_0^2}{1-\nu_0}\alpha_2 - \frac{15-60\nu_0+76\nu_0^2-26\nu_0^3}{2(1-2\nu_0)(1-\nu_0)^2}\alpha_1^2 + \right.$$
$$\left. + \frac{2-3\nu_0-\nu_0^2-7\nu_0^3+4\nu_0^4}{2(1-2\nu_0)(1-\nu_0)^2}\alpha_2^2 - \frac{\nu_0-10\nu_0^2+24\nu_0^3-10\nu_0^4}{(1-2\nu_0)(1-\nu_0)^2}\alpha_1\alpha_2\right]$$

(for D_2 it is necessary to interchange the subscripts 1 and 2),

$$
\begin{aligned}
D_k &= \frac{G_0 h_0^3}{12}\left[1 - \frac{1+5\nu_0}{2(1-\nu_0)}(\alpha_1+\alpha_2) - \right.\\
&\left. - \frac{9-34\nu_0+29\nu_0^2+10\nu_0^3}{4(1-2\nu_0)(1-\nu_0)^2}(\alpha_1^2+\alpha_2^2) - \frac{2-8\nu_0+4\nu_0^2+9\nu_0^3}{(1-2\nu_0)(1-\nu_0)^2}\alpha_1\alpha_2\right]\\
D_1\nu_2 &= \frac{E_0\nu_0 h_0^3}{12(1-\nu_0^2)}\left[1 - \frac{2\nu_0}{1-\nu_0}(\alpha_1+\alpha_2) - \right.\\
&\left. - \frac{3-12\nu_0+16\nu_0^2-2\nu_0^3}{2(1-2\nu_0)(1-\nu_0)^2}(\alpha_1^2+\alpha_2^2) - \frac{1-3\nu_0+3\nu_0^2+4\nu_0^3}{(1-2\nu_0)(1-\nu_0)^2}\alpha_1\alpha_2\right]
\end{aligned}
\tag{6.6}
$$

If the plate is stretched or compressed only in one direction, then it will bend as a tranversely isotropic one, with planes of isotropy perpendicular to the direction of extension, under longitudinal forces, $p = E_0 h\,\alpha(1 + 2\nu_0\alpha)$, acting on two opposite edges, [see (5.4)]. In this, α is the relative elongation of the plate; and $h = h_0(1 - \nu_0\alpha)$, if h_0 is the thickness of the plate before extension. The initial stress different from zero is

$$\sigma' = E_0\alpha(1 + 2\nu_0\alpha) \tag{6.7}$$

The rigidities necessary for the bending analysis of the plate, may be found by substitution of $\alpha_1 = \alpha$, $\alpha_2 = -\nu_0\alpha$, into (6.5), (6.6), to obtain

$$
\begin{aligned}
D_1 &= \frac{E_0 h_0^3}{12(1-\nu_0^2)}\left[1 - \nu_0(1-\nu_0)\alpha - \frac{15-45\nu_0+27\nu_0^2+24\nu_0^3-23\nu_0^4+4\,5\nu_0^5}{2(1-2\nu_0)(1-\nu_0)}\alpha^2\right]\\
D_3 &= \frac{E_0 h_0^3}{12(1-\nu_0^2)}\left[1 - (1+3\nu_0)\alpha + \frac{2-\nu_0-15\nu_0^2+18\nu_0^3-6\nu_0^4}{2(1-2\nu_0)(1-\nu_0)}\alpha^2\right.\\
D_k &= \frac{G_0 h_0^3}{12}\left[1 - \frac{1+5\nu_0}{2}\alpha - \frac{9-24\nu_0+13\nu_0^2+10\nu_0^3}{4(1-2\nu_0)}\alpha\right]\\
D_1\nu_2 &= \frac{E_0 h_0^3}{12(1-\nu_0^2)}\left[\nu_0 - 2\nu_0^2\alpha - \frac{3\nu_0-11\nu_0^2+14\nu_0^3-6\nu_0^4+2\nu_0^5}{2(1-2\nu_0)(1-\nu_0)}\alpha^2\right]
\end{aligned}
\tag{6.8}
$$

3. *Bending of a plate after application of shear.* At first we assume that bending is applied to a plate subsequently to pure shear. Let the edges of the plate be parallel to axes 1, 2 in Fig.2, that is in the principal directions for pure shear. The initial stresses are determined from (5.8). During a subsequent application of normal loading, the plate will bend as an orthotropic one, with the principal directions, parallel to its edges subjected to arbitrary forces $\sigma_1'h$ and $\sigma_2'h$, acting parallel to these edges, where σ_1' and σ_2' are taken from (5.8). The quantities, necessary for the determination of rigidities are found by substituting expressions (5.7) into (6.4):

$$D_1 = \frac{E_0 h^3}{12(1-\nu_0^2)}\left[1 - \frac{1+\nu_0}{2}s - \frac{7-24\nu_0+26\nu_0^2-7\nu_0^3}{4(1-2\nu_0)(1-\nu_0)}s^2\right] \quad (s = \alpha_2 - \alpha_1)$$

$$D_2 = \frac{E_0 h^3}{12(1-\nu_0^2)}\left[1 + \frac{1+\nu_0}{2}s - \frac{7-24\nu_0+26\nu_0^2-7\nu_0^3}{4(1-2\nu_0)(1-\nu_0)}s^2\right] \tag{6.9}$$

$$D_k = \frac{G_0 h^3}{12}\left[1 - \frac{3-4\nu_0}{4(1-2\nu_0)}s^2\right], \qquad D_2,\ \nu_1 = \frac{E_0\nu_0 h^3}{12(1-\nu_0^2)}\left[1 - \frac{2-7\nu_0+7\nu_0^2}{4(1-2\nu_0)(1-\nu_0)}s^2\right]$$

Now let us assume that two edges of a rectangular plate in a deformation field of pure shear, coincide with axes 1′, 2′, rotated relative to the principal directions 1, 2 of pure shear by the angle $\psi = \frac{1}{4}\pi - \frac{1}{2}\beta$ and tan $\beta = \frac{1}{2} s$, where s is the magnitude of simple shear (see Fig.2). The plate will undergo pure shear (see the remark at the end of Section 5). The stresses corresponding to this shear are found by formulas

$$\begin{aligned} \sigma_{11}' &= \sigma_1' \cos^2\varphi + \sigma_2' \sin^2\varphi = \tfrac{1}{4}(\lambda+\mu)\, s^2 \\ \sigma_{22}' &= \sigma_1' \sin^2\varphi + \sigma_2' \cos^2\varphi = \tfrac{1}{4}(\lambda+\mu)\, s^2 \\ \sigma_{12}' &= (\sigma_2' - \sigma_1') \sin\varphi\cos\varphi = \mu s \end{aligned} \tag{6.10}$$

where σ_1', σ_2' are taken from (5.8). Formulas (6.10) indicate that the deformation of pure shear cannot be caused by shear forces only; if it is to be significant, also normal forces should be applied to the edges of the plate, proportional to the square of the magnitude of shear, s (regarding the insufficiency of shear forces, see, for example, Green [12]). The rigidities of the plate, relative to axes 1, 2, are given by formulas (6.9). D_{11}', D_{22}', necessary for the determination of the bending rigidity, the torsional rigidity D_{66}' and the relative Poisson's ratio $\gamma_1 = D_{12}'/D_{22}'$, as well as the secondary rigidities D_{16}', D_{26}', absent for axes 1, 2, are determined by formulas [8]

$$\begin{aligned} D_{11}' &= D_1 \cos^4\varphi + 2D_3 \sin^2\varphi\cos^2\varphi + D_2 \sin^4\varphi \\ D_{22}' &= D_1 \sin^4\varphi + 2D_3 \sin^2\varphi\cos^2\varphi + D_2 \cos^4\varphi \\ D_{66}' &= D_k + (D_1 + D_2 - 2D_3)\sin^2\varphi\cos^2\varphi \end{aligned} \tag{6.11}$$

$$\gamma_1 = \frac{1}{D_{22}'}\left[D_2\nu_1 + (D_1 + D_2 - 2D_3)\sin^2\varphi\cos^2\varphi\right]$$

$$D_{16}' = \tfrac{1}{2}(D_2\sin^2\varphi - D_1\cos^2\varphi + D_3\cos 2\varphi)\sin 2\varphi$$

$$D_{26}' = \tfrac{1}{2}(D_2\cos^2\varphi - D_1\sin^2\varphi - D_3\cos 2\varphi)\sin 2\varphi$$

Substituting the expressions (6.9) and $\phi = \frac{1}{4}\pi - \frac{1}{2}\beta$ into (6.11), we obtain the rigidities, required for the analysis of a plate in bending:

$$\begin{aligned} D_{11}' &= \frac{E_0 h^3}{12(1-\nu_0^2)}\left[1 - \frac{3 - 9\nu_0 + 7\nu_0^2}{3(1-2\nu_0)(1-\nu_0)} s^2\right] \\ D_{22}' &= \frac{E_0 h^3}{12(1-\nu_0^2)}\left[1 - \frac{2 - 7\nu_0 + 8\nu_0^2 - 2\nu_0^3}{2(1-2\nu_0)(1-\nu_0)} s^2\right] \\ D_{66}' &= \frac{E_0 h^3}{12(1-\nu_0^2)}\left[\frac{1+\nu_0}{2} s - \frac{7 - 26\nu_0 + 33\nu_0^2 - 14\nu_0^3}{2(1-2\nu_0)(1-\nu_0)} s^2\right] \end{aligned} \tag{6.12}$$

$$\gamma_1 = \nu_0 - \tfrac{1}{2}(1-\nu_0)^2 s^2, \qquad D_{16}' = D_{26}' = \frac{E_0 h^3}{12(1-\nu_0^2)}\,\frac{1+\nu_0}{4}\, s$$

To check the calculations, we may use the relationships derived from (6.11)

$$D_{11}' + D_{22}' + 2D_{12}' = D_1 + D_2 + 2D_2\nu_1$$
$$D_{66}' - D_{12}' = D_k - D_2\nu_1 \quad (6.13)$$

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